The Tensor Rank of the Tripartite State $|W\rangle^{\otimes n}$

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Tensor rank refers to the number of product states needed to express a given multipartite quantum state. Its non-additivity as an entanglement measure has recently been observed. In this brief report, we estimate the tensor rank of multiple copies of the tripartite state $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$. Both an upper bound and a lower bound of this rank are derived. In particular, it is proven that the rank of $|W\rangle^{\otimes 2}$ is seven, thus resolving a previously open problem. Some implications of this result are discussed in terms of transformation rates between $|W\rangle^{\otimes n}$ and multiple copies of the state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.

With quantum entanglement being a proven asset to information processing and computational tasks, much effort has been devoted to quantifying it as a resource. One particular entanglement measure for a general pure state $|\psi\rangle \in H_1 \otimes H_2 \otimes ...H_n$ is the tensor rank which, denoted by $rk(|\psi\rangle)$, is defined as the minimum number r such that there exist states $|\phi_{ik}\rangle \in H_k$ ($1 \leq i \leq r$ and $1 \leq k \leq n$) for which

$$|\psi\rangle = \sum_{k=1}^{r} |\phi_{i1}\rangle|\phi_{i2}\rangle \cdots |\phi_{in}\rangle.$$

Tensor rank is a legitimate entanglement measure since it is a strictly non-increasing quantity under local operations (even probabilistically). Another key feature is that it provides a method of detecting entanglement: a multipartite pure state is entangled if and only if its tensor rank is larger than one. Further properties of the tensor rank and its application to quantum entanglement have been studied by Eisert and co-workers in Ref. [1] and [2]. For bipartite systems, tensor rank is equivalent to the so-called "Schmidt rank".

One way of partitioning a multipartite state space is as follows: two states are considered to be equivalent if and only if they can be reversibly converted from one to the other by operations belonging to the class of Stochastic Local Operations with Classical Communication (SLOCC). The ability to transform a state $|\psi\rangle$ to a state $|\phi\rangle$ with SLOCC is symbolically expressed as $|\psi\rangle \xrightarrow{\text{SLOCC}} |\phi\rangle$. It is well-known that in bipartite systems, $|\psi\rangle \xrightarrow{\text{SLOCC}} |\phi\rangle$ if and only if $rk(|\psi\rangle) \geq rk(|\phi\rangle)$. Consequently, SLOCC convertibility induces a total ordering among bipartite pure states that can be completely characterized by the Schmidt rank. Another nice property of the Schmidt rank is that it is additive in the sense that $\log_2(rk(|\psi\rangle^{\otimes n})) = n \cdot \log_2(rk(|\psi\rangle))$. In contrast, the general multipartite tensor rank is insufficient for determining SLOCC equivalence [3], and it is not an additive quantity [4].

The main purpose of this brief report is to evaluate the tensor rank of the tripartite state $|W\rangle^{\otimes n}$. As the rank is unchanged by overall scalar multiplications, from now on we will work with unnormalized states. Dür *et al.* [5] were the first to observe that within three qubit systems, there exist two distinct equivalence classes of genuinely tripartite entangled states, one represented by the state $|GHZ\rangle$ and the other represented by $|W\rangle$. While single copy transformations are not possible between $|W\rangle$ and $|GHZ\rangle$, a natural question is: how many copies of $|W\rangle$ are needed to obtain a single $|GHZ\rangle$ by SLOCC, and vice versa? More precisely, what are the lowest possible ratios $\frac{m}{n}$ for the respective transformations $|GHZ\rangle^{\otimes m} \xrightarrow{\text{SLOCC}} |W\rangle^{\otimes n}$ and $|W\rangle^{\otimes m} \xrightarrow{\text{SLOCC}} |GHZ\rangle^{\otimes n}$? As noted above, the tensor rank is monotonically decreasing under SLOCC. Hence a way of evaluating these ratios is to calculate the tensor rank for the above multiple-copy states.

In Ref. [4], an eight-term product state expansion for $|W\rangle^{\otimes 2}$ was found. This indicates that $rk(|W\rangle^{\otimes 2}) \leq 8$, and the transformation $|GHZ\rangle^{\otimes 3} \xrightarrow{\text{SLOCC}} |W\rangle^{\otimes 2}$ is feasible. In this brief report, we determine the exact value $rk(|W\rangle^{\otimes 2}) = 7$. As an immediate consequence, we have the improved upper bound of $rk(|W\rangle^{\otimes n}) \leq 7^{\lceil n/2 \rceil}$. On the other hand, we find a lower bound of $rk(|W\rangle^{\otimes n}) \geq 2^{n+1} - 1$. It is obvious that this lower bound is tight when $n = 2$.

In the following, we will use $\text{Tr}_i(\cdot)$ to stand for a partial trace with respect to subsystem i, and the support of a positive operator will refer to its range. We will need the following lemma from Ref. [4], which gives an alternative

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description of tensor rank.

Lemma 1. Suppose $|\psi\rangle \in H_1 \otimes \cdots \otimes H_n$. For an arbitrarily chosen i, $rk(|\psi\rangle)$ equals the minimum number of product states in $H_1 \otimes \cdots \otimes H_{i-1} \otimes H_{i+1} \otimes \cdots \otimes H_n$ whose linear span contains the support of $\mathrm{Tr}_i(|\psi\rangle\langle\psi|)$.

For the state $|W\rangle^{\otimes 2} \in H_A \otimes H_B \otimes H_C$, let $\rho = \mathrm{Tr}_A(|W^{\otimes 2}\rangle\langle W^{\otimes 2}|)$, the support of ρ is spanned by the states

$$\{|00\rangle, |01\rangle + |10\rangle, |02\rangle + |20\rangle, |03\rangle + |12\rangle + |21\rangle + |30\rangle\},$$

for $0 \le i, j < 4$, $|ij\rangle = |i\rangle_B |j\rangle_C$, $|i\rangle_B$ and $|j\rangle_C$ denote the two-qubits' state of B-part and C-part, respectively.(For simplicity, here we assume that $|0\rangle = |00\rangle$, $|1\rangle = |01\rangle$, $|2\rangle = |10\rangle$, and $|3\rangle = |11\rangle$).

One can easily check that this set is contained in the linear span of

$$\{|00\rangle, |03\rangle, |30\rangle, |\Psi_1\rangle|\Psi_1\rangle, |\Psi_2\rangle|\Psi_2\rangle, |\Psi_3\rangle|\Psi_4\rangle, |\Psi_4\rangle|\Psi_4\rangle\}$$

where $|\Psi_1\rangle = |0\rangle + |1\rangle + |2\rangle$, $|\Psi_2\rangle = |0\rangle + |1\rangle - |2\rangle$, $|\Psi_3\rangle = |1\rangle + |2\rangle$, and $|\Psi_4\rangle = |1\rangle - |2\rangle$. Indeed, we have:

$$\begin{aligned}
|00\rangle + |01\rangle + |10\rangle &= (|\Psi_1\rangle|\Psi_1\rangle - |\Psi_3\rangle|\Psi_3\rangle \\
&\quad + |\Psi_2\rangle|\Psi_2\rangle - |\Psi_4\rangle|\Psi_4\rangle)/2, \\
|02\rangle + |20\rangle &= (|\Psi_1\rangle|\Psi_1\rangle - |\Psi_3\rangle|\Psi_3\rangle \\
&\quad - |\Psi_2\rangle|\Psi_2\rangle + |\Psi_4\rangle|\Psi_4\rangle)/2, \\
|12\rangle + |21\rangle &= (|\Psi_3\rangle|\Psi_3\rangle - |\Psi_4\rangle|\Psi_4\rangle)/2.
\end{aligned}$$

Lemma 1 implies that $rk(|W\rangle^{\otimes 2}) \le 7$. Using the above equalities one obtains a seven term decomposition:

$$\begin{aligned}
|W\rangle^{\otimes 2} =\ & |300\rangle + |030\rangle + |00\rangle(|3\rangle - |2\rangle) \\
& + |\Psi_1\rangle|\Psi_1\rangle(|0\rangle - |1\rangle - |2\rangle)/2 \\
& + |\Psi_2\rangle|\Psi_2\rangle(|1\rangle - |0\rangle - |2\rangle)/2 \\
& + |\Psi_3\rangle|\Psi_3\rangle(|1\rangle + |2\rangle))/2 + |\Psi_4\rangle|\Psi_4\rangle(|2\rangle - |1\rangle)/2.
\end{aligned}$$

We now proceed to prove a lower bound for the tensor rank of n copies of $|W\rangle$. By Lemma 1, it is sufficient to evaluate the minimum number of product states whose linear span contains the set

$$\{|00\rangle, |01\rangle + |10\rangle\}^{\otimes n} = \{|\varphi_i\rangle|0 \le i \le 2^n - 1\}.$$

Without loss of generality, assume that $|\varphi_0\rangle = |0\rangle^{\otimes n}$ and $|\varphi_{2^n-1}\rangle = (|01\rangle + |10\rangle)^{\otimes n}$. The essential piece in the following lemma is the fact that the state

$$|\varphi_{2^n-1}\rangle + \sum_{i=0}^{2^n-2} \alpha_i |\varphi_i\rangle$$

always has a Schmidt rank of 2^n for any $\alpha_i \in \mathbb{C}$. This can be easily verified by calculating the matrix rank after taking a partial trace.

Lemma 2. $rk(|W\rangle^{\otimes n}) \ge 2^{n+1} - 1$.

Proof: Assume that there exist $2^{n+1} - 2$ product states $\{|\xi_i\rangle|0 \le i \le 2^{n+1} - 3\}$ whose linear span contains $\{|\varphi_i\rangle|0 \le i \le 2^n - 1\}$. Without loss of generality, let $|\xi_0\rangle = |\varphi_0\rangle$. For $1 \le i \le 2^n - 1$, put

$$|\varphi_i\rangle = \sum_{k=0}^{2^{n+1}-3} \beta_{ik} |\xi_k\rangle.$$

Since the matrix $B = (\beta_{ik})$ for $1 \le i \le 2^n - 2$ and $0 \le k \le 2^{n+1} - 3$ has rank $2^n - 2$, there exist $2^n - 2$ linear independent columns $l_1, l_2 \cdots l_{2^n-2}$. Then, one can find $2^n - 2$ complex number $\delta_1, \delta_2 \cdots \delta_{2^n-2}$ such that $\mu_{l_i} = 0$ for all $1 \le i \le 2^n - 2$ and

$$|\varphi_{2^n-1}\rangle + \sum_{i=1}^{2^n-2} \delta_i |\varphi_i\rangle = \sum_{i=0}^{2^{n+1}-3} \mu_{l_i} |\xi_i\rangle.$$

Consequently, we see that

$$|\varphi_{2^n-1}\rangle + \sum_{i=1}^{2^n-2} \delta_i |\varphi_i\rangle + \mu_0 |\varphi_0\rangle = \sum_{i=1}^{2^{n+1}-3} \mu_i |\xi_i\rangle$$

is a $2^{n+1} - 3 - (2^n - 2) = 2^n - 1$ product state expansion. This, however, is impossible since $|\varphi_{2^n-1}\rangle$ has a Schmidt rank of 2^n. $\blacksquare$

In light of our seven term construction for $|W\rangle^{\otimes 2}$, we now have the following

Lemma 3. $rk(|W^{\otimes 2}\rangle) = 7$.

Combining the above lemmas as well as some of their immediate consequences, we obtain the main results of this paper and state them in the following theorem. Part (b) relies on the facts that $rk(|GHZ\rangle^{\otimes n}) = 2^n$ and that a non-increase in tensor rank is sufficient for convertibility from $|GHZ\rangle^{\otimes n}$.

Theorem 1.

$$(a) \quad 2^{n+1} - 1 \le rk(|W\rangle^{\otimes n}) \le \begin{cases} 7^{n/2} & \text{even } n \\ 3 \cdot 7^{\frac{n-1}{2}} & \text{odd } n, \end{cases}$$

$$(b) \quad \frac{m}{n+1} \ge \tfrac{1}{2} \log_2 7 \Rightarrow |GHZ\rangle^{\otimes m} \xrightarrow{\text{SLOCC}} |W\rangle^{\otimes n},$$

$$(c) \quad |W\rangle^{\otimes m} \xrightarrow{\text{SLOCC}} |GHZ\rangle^{\otimes n} \Rightarrow \frac{n}{m+1} \le \tfrac{1}{2} \log_2 7.$$

Let $N = 2^n$, part (a) of the above theorem can be restated as follows: $2N - 1 \le rk(|W\rangle^{\otimes \log_2 N}) \le O(N^\nu)$ where $\nu = \frac{\log_2 7}{2}$. There exists an interesting similarity between this and the bounds on the tensor rank of the state $|\Phi^3\rangle^{\otimes \log_2 N}$ where $|\Phi^3\rangle = |\Phi\rangle_{AB} |\Phi\rangle_{AC} |\Phi\rangle_{BC}$ and $|\Phi\rangle_{ij} = |0_i 0_j\rangle + |1_i 1_j\rangle$. In Ref. [4] it was shown that $\frac{5}{2} N^2 - 3N \le rk(|\Phi^3\rangle)^{\otimes \log_2 N} \le O(N^\omega)$ where $\omega = 2.36$. It turns out that ω corresponds to the so-called "exponent for matrix multiplication" which is the smallest real number such that an algorithm exists for multiplying two $N \times N$ matrices using $O(N^\omega)$ multiplications. Extensive work has been devoted to determining the exact value

of ω and most researchers hypothesize its value to be two. In a similar manner, we can define the "exponent for the W-state" as the smallest real number ν such that $rk(|W\rangle^{\otimes \log_2 N}) \leq O(N^\nu)$. We conjecture that $\nu = 1$, and if this were true, then for any $\epsilon > 0$, there would exist some n such that $|GHZ\rangle^{\otimes \lfloor n(1+\epsilon) \rfloor} \xrightarrow{\text{SLOCC}} |W\rangle^{\otimes n}$. It remains an open challenge to verify whether or not this speculation is correct.

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[1] J. Eisert and H. J. Briegel, Phys. Rev. A **64**, 022306 (2001).

[2] M. Hein, J. Eisert, and H. J. Briegel, Phys. Rev. A **69**, 062311 (2004).

[3] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, Phys. Rev. A **65**, 052112 (2002).

[4] E. Chitambar, R. Duan, and Y. Shi, Phys. Rev. Lett. **101**, 140502 (2008).

[5] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).